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ON A QUESTION OF GRINBLAT

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Abstract. We prove the consistency of: there is a κ-complete ideal on κ for some $\kappa < 2^{\aleph_0}$ such that the Boolean algebra $\mathcal{P}(\kappa)/I$ is σ-centered and there are Q-sets of reals.

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§1

In set theoretic language, Grinblat has been asking for some time
<u>1.1 Problem</u>: Is it consistent with ZFC that:

- (a) there is an $\aleph_1$-complete ideal I on some $\kappa < 2^{\aleph_0}$ such that $\mathscr{P}(\kappa)/I$ is centered
- (b) there is a Q-set.

We answer positively.

1.2 Claim. *Assume that $\kappa < \chi = \chi^\kappa$ and κ is measurable, say D is a normal ideal on κ and I is the dual ideal.*

<u>Then</u> for some c.c.c. forcing notion P of cardinality χ we have in V^P:

$(*)(i)$ $2^{\aleph_0} = \chi$

(ii) $\mathrm{MA}_{<\kappa,<\mathrm{cf}(\chi)}(\sigma\text{-centered})$ *holds (i.e.* MA *for forcing notions of cardinality $< \kappa$ and $< \mathrm{cf}(\chi)$ dense open subsets hence every set of reals of cardinality $< \kappa$ is a Q-set*

(iii) *the Boolean algebra $\mathscr{P}(\kappa)/I$ is centered, i.e. $\mathscr{P}(\kappa)\backslash I^{V^P}$ is the union of countable many directed sets (where*
$I^{V^P} = \{A \in V^P : A \subseteq \kappa$ and A is included in some member of $I\}$.

1.3 Remark. 1) Why the "hence" in (ii)? As for $X \subseteq Y \subseteq {}^\omega 2$ the natural forcing Q of adding subtrees $\underset{\sim}{T}_n \subseteq {}^{\omega>}2$ for $n < \omega$ such that $\bigcup\limits_{n<\omega} \lim(\underset{\sim}{T}_n) \cap Y = X$ is σ-centered of cardinality $\le |Y| + \aleph_0$ and it is enough to find a directed $G \subseteq Q$ intersecting $|Y| + \aleph_0$ dense subsets. E.g. we can use:

$$Q = \Big\{ p = (\bar t, \bar u) \text{ :for some } n = n(p) < \omega \text{ we have :}$$

$$\begin{aligned} &(a) \quad \bar t = \langle t_\ell : \ell < n \rangle, \text{ each } t_\ell \text{ has the form } T \cap {}^{m_\ell(p)\ge}2, T \\ &\qquad \text{a perfect subtree of } {}^{\omega>}2, m_\ell(p) < \omega, \\ &(b) \quad \bar u = \langle u^*_\ell : \ell < n(p) \rangle, u'_\ell \subseteq X \text{ is finite} \\ &(d) \quad \text{if } \ell < n(p), n \in u_\ell \text{ then } \eta \restriction m_\ell(p) \in t_\ell \Big\}. \end{aligned}$$

The order is natural $p \le q$ iff $n(p) \le n(q)$ & $\bigwedge\limits_{\ell<n(p)} [t^p_\ell \subseteq t^q_\ell$,
$m_\ell(p) \le m_\ell(q)$ & $t^p_\ell = t^q_\ell \cap {}^{m_\ell)p)\ge}2]$ & $\bigwedge\limits_{\ell<n(p)} u^p_\ell \subseteq u^q_\ell$.

Let for $\eta \in X, \mathscr{I}_\eta$ be $\{p \in Q : \eta \in u^p_\ell$ for some $\ell < n(p)\}$ and for $x \in Y \backslash X, n < \omega$
let $\mathscr{I}_{x,n} = \{p \in Q : n < n(p)$ and $\eta \restriction m_n(p) \notin t^p_n\}$

Proof of 1.2. Let $\bar{Q} = \langle P_i, Q_j : i \le \chi, j < \chi \rangle$ be a FS iteration, in $V^{P_i}, \underset{\sim}{Q}_j$ is a σ-centered forcing notion of cardinality $< \kappa$ and its set of elements is an ordinal $< \kappa$, and any such forcing notion appear unboundedly often, more exactly, if $i_0 < \chi, \underset{\sim}{Q}$ is a P_{i_0}-name of a forcing notion with a set of elements (forced to be) $\underset{\sim}{\alpha}_Q < \kappa$, then for χ many (hence unboundedly many) $j \in (i, \chi)$ we have: $\Vdash_{P_j}$ "if $\underset{\sim}{Q}$ is σ-centered then $\underset{\sim}{Q} \cong \underset{\sim}{Q}_j$".

As each $\underset{\sim}{Q}_j$ is σ-center (in V^{P_j}) there is $\underset{\sim}{\bar{f}} = \langle \underset{\sim}{f}_j : j < \chi \rangle$ such that $\Vdash_{P_j}$ "$\underset{\sim}{f}_j$ is a function from $\underset{\sim}{Q}_j$ to ω such that each $\{p \in \underset{\sim}{Q}_j : \underset{\sim}{f}_j(p) = n\}$ is directed".

So $P_{1,i} =: \{p \in P_i :$ if $j \in \operatorname{Dom}(p)$ then $p \restriction j$ forces a value to $\underset{\sim}{f}_j(p(j))$ and a vlue to $p(j)\}$ is a dense subset of P_i. Now clearly clauses (i), (ii) of $(*)$ holds in V^P. As D is a normal ultrafilter on κ there is a transitive class M such that $M^\kappa \subseteq M$ and there is an elementary embedding $\mathbf{j}$ from V to M with critical ordinal κ such that $D = \{A : A \in V, A \subseteq \chi$ and $\kappa \in \mathbf{j}(A)\}$. Let $\mathbf{j}(\bar{Q})$ be $\bar{Q}' = \langle P'_i, \underset{\sim}{Q}'_j : i \le \mathbf{j}(\kappa), j < \mathbf{j}(\chi) \rangle$ and $\underset{\sim}{\bar{f}}' = \mathbf{j}(\underset{\sim}{\bar{f}}' = \langle f'_j : j < \mathbf{j}(\kappa) \rangle$, so M "thinks" that $(\bar{Q}', \underset{\sim}{\bar{f}}')$ satisfies all the properties listed above, but in V it relates all of those properties, though not $\mathbf{j}(\kappa)$.

Let $P^* = \{\mathbf{j}(p) : p \in P'_\chi\}$, so it is well known that $P^* \lessdot P'_{\mathbf{j}(\chi)}$ and the completion of the Boolean algebra corresponding to $P'_{\mathbf{j}(\chi)}/P^*$ is isomorphic to $\mathscr{P}(\kappa)/I^{V^P}$, so it is enough to prove that $\Vdash_{P^*}$ "$P'_{\mathbf{j}(\chi)}/P^*$ is σ-centered" (in V^{P^*}, which is the same as V^{P_χ}). Note also the $P'_{\mathbf{j}(\chi)}$ hence $P'_{\mathbf{j}(\chi)}/P^*$ has cardinality $\le |P_{1,\chi}|^\kappa = \chi^\kappa = \chi$.

Now the point is that we can reorder the iteration $\bar{Q}'$: first do $\langle Q_{\mathbf{j}(j)} : j < \chi \rangle$ and then the rest, as each Q_j depends on $< \kappa, j' < j$ and this set is not extended by $\mathbf{j}$.

Note first that this suffices as the limit of FS iteration of σ-centered forcing notion each of cardinality $\le 2^{\aleph_0}$ and length $< (2^{\aleph_0})^+$ (in $V^{P_{1,\chi}}$!) is σ-centered.

Second, this reordering is possible.

[Why? The set of elements of $\underset{\sim}{Q}_j$ is $\underset{\sim}{\alpha}_{Q_j}$ and $\underset{\sim}{\alpha}_{Q_j}$ is a P'_j-name of an ordinal $< \kappa$ and P'_j satisfies the c.c.c. hence for some $\alpha^*_j < \kappa$ we have $\Vdash_{P^*_{1,j}}$ "$\underset{\sim}{\alpha}^*_{Q_j} \le \alpha^*_j$" so $\underset{\sim}{Q}_j$ is a subset of $\alpha^*_j \times \alpha^*_j$. For any $\beta, \gamma < \alpha^*$, there is a maximal antichain $\mathscr{I}_{j,\beta,\gamma}$ of $P^*_{1,j}$ of conditions forcing "$\underset{\sim}{Q}_j \models \beta \le \gamma^{(*)}$, so $\beta \in \underset{\sim}{Q}_j, \gamma \in \underset{\sim}{Q}_j$ or forcing its negation. We choose $\bar{I} = \langle \mathscr{I}_{j,\beta,\gamma} : j < \chi, \beta, \gamma < \alpha^*_j \rangle$. Let $A_j = \bigcup\limits_{\beta,\gamma < \alpha^*_j} \bigcup\limits_{p \in \mathscr{I}_{j,\beta,\gamma}}$, $\operatorname{Dom}(p)$, so $|A_j| < \kappa$ and call $A \subseteq \chi$ $\bar{Q}$-closed if $(\forall j \in A)(A_j \subseteq A)$. Now in M we can compute $j(\bar{I})$ hence A^M_j for $j < \mathbf{j}(\chi)$, now easily $A^M_{\mathbf{j}(j)} = \{\mathbf{j}(i) : i \in A_j\}$ as $|A_j| < \kappa$, so this reordering is O.K.]

So we are done. $\square_{1.2}$